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Further Extensions of a Legendre Function Integral

By Henry E. Fettis*

Abstract. The integral

$$\int_z^1 \left(\frac{1-t}{2}\right)^{\beta-1}\left(\frac{1-t}{1+t}\right)^{\mu/2}\ln\left(\frac{1-t}{2}\right)P_{\nu-1}^{\mu}(t)\,dt$$

is evaluated as a hypergeometric function for arbitrary values of "ν", "μ", $-1 \leqslant z \leqslant 1$, and $\mathrm{Re}(\beta) > 0$.

In this paper, we obtain expressions for integrals of the type

$$\int_z^1\left(\frac{1-t}{2}\right)^{\beta-1}\left(\frac{1-t}{1+t}\right)^{\mu/2}\ln\left(\frac{1-t}{2}\right)P_\nu^\mu(t)\,dt, \tag{1}$$

where $P_\nu^\mu(z)$ is the Legendre function, $-1 \leqslant z \leqslant 1$, and $\mathrm{Re}(\beta) > 0$. The relations thus obtained include as special cases results given previously by Blue [1], Gautschi [2], Ainsworth and Liu [3], [6], Gatteschi [7], and the present author [4].

The starting point of the development is the beta-transform for the hypergeometric function ${}_pF_q(z)$ as given in [5, Art. 5.2.3]:

$$\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-1}{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\zeta t\right)dt = {}_{p+1}F_{q+1}\left(\begin{matrix}\beta,\alpha_p\\ \beta+\gamma,\rho_q\end{matrix}\middle|\zeta\right) \tag{2}$$

which, with an obvious change of variables, may be written

$$\begin{aligned}&\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\int_z^1 (1-x)^{\beta-1}(x-z)^{\gamma-1}{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\frac{1-x}{2}\right)dx\\ &\qquad = (1-z)^{\beta+\gamma-1}{}_{p+1}F_{q+1}\left(\begin{matrix}\beta,\alpha_p\\ \beta+\gamma,\rho_q\end{matrix}\middle|\frac{1-z}{2}\right),\end{aligned} \tag{3}$$

and since

$$\Gamma(1-\mu)P_{\nu-1}^\mu(x) = \left(\frac{1+x}{1-x}\right)^{\mu/2}{}_2F_1\left(\begin{matrix}\nu,1-\nu\\ 1-\mu\end{matrix}\middle|\frac{1-x}{2}\right), \tag{4}$$

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*Deceased December 15, 1984.

(3) gives

$$\text{(5)}\qquad \begin{aligned}\Gamma(1-\mu)\int_z^1 \left(\frac{1-x}{2}\right)^{\beta-1}\left(\frac{x-z}{2}\right)^{\gamma-1}\left(\frac{1-x}{1+x}\right)^{\mu/2} P_{\nu-1}^{\mu}(x)\,dx \\ = 2\left(\frac{1-z}{2}\right)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\,{}_3F_2\left(\begin{matrix}\beta,\nu,1-\nu\\ \beta+\gamma,1-\mu\end{matrix}\middle|\frac{1-z}{2}\right).\end{aligned}$$

Equation (5) may be specialized in a number of ways. For example, if $\beta = 1$, we get

$$\text{(6a)}\qquad \begin{aligned}\Gamma(1-\mu)\int_z^1 \left(\frac{x-z}{2}\right)^{\gamma-1}\left(\frac{1-x}{1+x}\right)^{\mu/2} P_{\nu-1}^{\mu}(x)\,dx \\ = \frac{2}{\gamma}\left(\frac{1-z}{2}\right)^{\gamma}{}_3F_2\left(\begin{matrix}1,\nu,1-\nu\\ 1+\gamma,1-\mu\end{matrix}\middle|\frac{1-z}{2}\right);\end{aligned}$$

$$\text{(6b)}\qquad \begin{aligned}\Gamma(1-\mu)\int_{-1}^1 \left(\frac{1+x}{2}\right)^{\gamma-1}\left(\frac{1-x}{1+x}\right)^{\mu/2} P_{\nu-1}^{\mu}(x)\,dx \\ = \frac{2}{\gamma}\,{}_3F_2\left(\begin{matrix}1,\nu,1-\nu\\ 1+\gamma,1-\mu\end{matrix}\middle|1\right),\end{aligned}$$

and, if, in addition, $\mu = 0$, the ${}_3F_2$ function reduces to an ${}_2F_1$, and we obtain the following:

$$\text{(7a)}\qquad \int_z^1\left(\frac{x-z}{2}\right)^{\gamma-1} P_{\nu-1}(x)\,dx = \frac{2}{\gamma}\left(\frac{1-z}{2}\right)^{\gamma}{}_2F_1\left(\begin{matrix}\nu,1-\nu\\ 1+\gamma\end{matrix}\middle|\frac{1-z}{2}\right);$$

$$\text{(7b)}\qquad \begin{aligned}\int_{-1}^1\left(\frac{x+1}{2}\right)^{\gamma-1} P_{\nu-1}(x)\,dx \\ = \frac{2}{\gamma}\,{}_2F_1\left(\begin{matrix}\nu,1-\nu\\ 1+\mu\end{matrix}\middle|1\right) = \frac{2[\Gamma(\gamma)]^2}{\Gamma(\gamma+\nu)\Gamma(\gamma-\nu+1)},\end{aligned}$$

the latter being equivalent to Eq. (3) of [2].

Similarly, by taking $\gamma = 1$ in (5) we get

$$\text{(8a)}\qquad \begin{aligned}\Gamma(1-\mu)\int_z^1 \left(\frac{1-x}{2}\right)^{\beta-1}\left(\frac{1-x}{1+x}\right)^{\mu/2} P_{\nu-1}^{\mu}(x)\,dx \\ = \frac{2}{\beta}\left(\frac{1-z}{2}\right)^{\beta}{}_3F_2\left(\begin{matrix}\beta,\nu,1-\nu\\ 1+\beta,1-\mu\end{matrix}\middle|\frac{1-z}{2}\right);\end{aligned}$$

$$\text{(8b)}\qquad \Gamma(1-\mu)\int_{-1}^1 \left(\frac{1-x}{2}\right)^{\beta-1}\left(\frac{1-x}{1+x}\right)^{\mu/2} P_{\nu-1}^{\mu}(x)\,dx = \frac{2}{\beta}\,{}_3F_2\left(\begin{matrix}\beta,\nu,1-\nu\\ 1+\beta,1-\mu\end{matrix}\middle|1\right)$$

and it appears that, in general, (8a) and (8b) cannot be reduced or simplified except in the special case where $\mu = 0$, $z = -1$, and ν is an integer, n, in which instance, the integrals in (7b) and (8b) differ only in sign, leading to the relation

$$\text{(9)}\qquad \frac{1}{\beta}\,{}_3F_2\left(\begin{matrix}\beta,n,1-n\\ 1+\beta,1\end{matrix}\middle|1\right) = \frac{(-1)^{n+1}[\Gamma(\beta)]^2}{\Gamma(\beta+n)\Gamma(\beta-n+1)},$$

a result also obtainable directly from formula 5.2.4(2) of [5].

On the other hand, if $\nu = n$ and μ is not a positive integer, the functions $P_{n-1}^{\mu}(x)$ and $P_{n-1}^{-\mu}(x)$ differ only by a numerical factor;

$$P_{n-1}^{-\mu}(-x) = (-1)^n \frac{\Gamma(n-\mu)}{\Gamma(n+\mu)} P_{n-1}^{\mu}(x) \tag{10}$$

which leads to the following, and not immediately evident, relation:

$$\begin{aligned}(-1)^{n-1}[(1-\mu)_{n-1}]\, {}_3F_2\left(\begin{matrix}\beta, n, 1-n\\ 1+\beta, 1-\mu\end{matrix}\middle| 1\right)\\ = [(1+\mu)_{n-1}]\, {}_3F_2\left(\begin{matrix}1, n, 1-n\\ 1+\beta, 1+\mu\end{matrix}\middle| 1\right).\end{aligned} \tag{11}$$

The corresponding integrals involving the logarithmic term are obtained by differentiation with respect to "β" (or "γ"). However, this can, apparently, only be done explicitly in two special cases, one being that where the integral is expressible by gamma-functions, as in (7b). In this instance, the result becomes the one given by Gautschi and the present author ([2], [4]), and previously for integer "ν" by Ainsworth and Liu [3]. (Cf. [2, Eq. (1)].) The second case where explicit expressions are obtainable for the logarithmic integrals occurs when $\gamma = 1$ (Eqs. (8a), (8b)), since then we have

$$\begin{aligned}-\beta^2 \frac{\partial}{\partial\beta}\left\{\frac{1}{\beta}\, {}_3F_2\left(\begin{matrix}\beta, \nu, 1-\nu\\ 1+\beta, 1-\mu\end{matrix}\middle| \frac{1-z}{2}\right)\right\}\\ = {}_4F_3\left(\begin{matrix}\beta, \beta, \nu, 1-\nu\\ 1+\beta, 1+\beta, 1-\mu\end{matrix}\middle| \frac{1-z}{2}\right),\end{aligned} \tag{12}$$

the result being a generalization of one previously found by Ainsworth [6] for $z = -1$, $\mu = 0$, and integer values of ν and β. Equation (12) also provides an explicit expression for the ${}_4F_3$ function when $z = -1$, $\mu = 0$. Further, the above idea can be extended to the case where γ is a positive integer, m, by noting that expansion of the term $(1-t)^{m-1}$ in the beta-transform (2) by the binomial theorem leads to the result**

$$\begin{aligned}\frac{1}{\beta}\, {}_3F_2\left(\begin{matrix}\beta, a, b\\ \beta+m, c\end{matrix}\middle| z\right)\\ = \frac{\Gamma(\beta+m)}{\Gamma(\beta+1)} \sum_{n=0}^{m-1} (-1)^m \binom{m-1}{n} \left(\frac{1}{\beta+n}\right) {}_3F_2\left(\begin{matrix}\beta+n, a, b\\ \beta+n+1, c\end{matrix}\middle| z\right).\end{aligned} \tag{13}$$

Additional observations are that, although (5), (6) and (8) were originally restricted to noninteger values of "μ", they are easily modified to include integer values of this parameter by applying the limiting expression given by Eq. 6.3(12) of [5] to the hypergeometric function in (5), and that with the aid of the relation

$$(\sin \nu\pi) Q_{\nu-1}^{\mu}(x) = \frac{\pi}{2} \cos(\mu-\nu)\pi \frac{\Gamma(\nu+\mu)}{\Gamma(\nu-\mu)} P_{\nu-1}^{-\mu}(x) + \cos(\mu\pi) P_{\nu-1}^{\mu}(-x), \tag{14}$$

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(6b) and (8b) may be combined to give, for noninteger "ν" and "μ", the following:

$$\sin(\nu\pi)\int_{-1}^{1}\left(\frac{1-t}{2}\right)^{\beta-1}\left(\frac{1-t}{1+t}\right)^{\mu/2}Q_{\nu-1}^{\mu}(t)\,dt$$

(15)
$$=\frac{\pi}{\beta}\cos(\mu+\nu)\pi\left\{\frac{\Gamma(\nu+\mu)}{\Gamma(1+\mu)\Gamma(\nu-\mu)}\,{}_3F_2\left(\begin{matrix}1,\nu,1-\nu\\1+\beta,1+\mu\end{matrix}\middle|1\right)\right\}$$
$$+\frac{\cos(\mu\pi)}{\Gamma(1-\mu)}\left\{{}_3F_2\left(\begin{matrix}\beta,\nu,1-\nu\\1+\beta,1-\mu\end{matrix}\middle|1\right)\right\}.$$

Finally, it may be noted that, since

$$P_n^{(p,q)}(2z-1)=(-1)^n\frac{(q+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,p+q+n+1\\q+1\end{matrix}\middle|z\right), \tag{16}$$

the beta-transform (3) also provides a simple means of obtaining results relating to the Jacobi polynomials $P_n^{(p,q)}(z)$ which are more general than some of those given in [7].** For example, Eq. (1.3) (loc. cit.) may be generalized in this way to

(17)
$$(-1)^n\frac{n!}{(q+1)_n}\int_{-1}^{z}(z-t)^{\alpha}(1+t)^{\beta+u}P_n^{(p,q)}(2t-1)\,dt$$
$$=(1+z)^{\alpha+\beta-1}\,{}_3F_2\left(\begin{matrix}-n,p+q+n+1,\beta+u+1\\q+1,\alpha+\beta+u+2\end{matrix}\middle|\frac{1+z}{2}\right)$$

which, in the special case where $z=1$, $p=\alpha$, $q=\beta$, can, with the aid of formula 5.2.4(2) of [5], be reduced to Eq. (1.5) of the first cited paper.

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